QS1 POLYNOMIALS.

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***Abstract**. The author earlier developed new classes of quasi-spectral polynomials, and the study presents new findings about these classes for the efficient resolution of mathematical physics problems. By examining the approximation behavior of Fourier series by systems of quasispectral polynomials and the accompanying order of approximation, we explore the potential for retrieving information about functions that are solutions of boundary value problems. This work proves that the function, which in practice is the Sobolev space solution of the boundary value problem, can be reconstructed with the same accuracy in the base space of all square summable functions as it could be reconstructed if it were explicitly given.*

***Keywords**: quasispectral polynomials, Fourier series, spectral methods, approximation methods, Sobolev spaces, orthogonal polynomials, classical polynomials.*

One of the major ways that computers solve hydrodynamic and gas dynamics problems is through the use of spectral approaches. Fourier series methods, Galerkin methods, the Lanczos tau method, Dzyadyk's approximation method (a-method) [1], and other techniques are examples of spectral methods. Classic orthogonal polynomials, such as Hermit, Laguerre, Jacobi, and especially Legendre and Chebyshov polynomials of the first and second kinds, are frequently used as basis functions in these techniques.

The author [2-4] proposed to use classes of quasispectral polynomials of the first kind $K_i^\circ(x), i = 1,\dots,2n+2$ and the second kind $K_i^\cdot(x), i = 0,\dots,2n+1$ as basic functions which depend of the natural parameter n. Polynomials $K_i^\circ, i = 2n$, which we call interior polynomials are orthogonal in the metric of $L_2[-1,1]$ space:

$$\left(K_i^\circ, K_j^\circ\right) = \int_{-1}^{1} K_i^\circ K_j^\circ dx = 0, i \neq j, \left\|K_i^\circ\right\| = \sqrt{\left(K_i^\circ, K_i^\circ\right)} = 1$$

simultaneously with its first derivatives $D = d/dx$, i.e. it means that

$$\left(DK_i^\circ, DK_j^\circ\right) = 0, i \neq j, (DK_i^\circ, K_i^\circ) = \|DK_i^\circ\|^2 = \lambda_i^\circ > 0.$$

Because of the previous facts, the interior quasispectral polynomials are orthogonal in the Hilbert space H^1 endowed with a inner product and a norm of the form

$$(u,v)_1 = \int_{-1}^{1}(DuDv + cuv)dx, \|u\|_1 = \sqrt{(u,u)_1}, \tag{1}$$

where $u = u(x), v = v(x)$, $c > 0$– some number. The subspace of the function space H^1 with zero boundary conditions will be denoted by H_0^1. It is introduced here as a closure in the metric H^1 of the set of all polynomials with zero boundary conditions of the first kind.

Each of the polynomials of the second kind $K_i^\bullet, i = 1,\dots,2n$ coincides with the derivative of the corresponding polynomial K_i° up to a certain factor:

$$DK_i^\circ = \sqrt{\lambda_i^\circ} K_i^\bullet. \tag{2}$$

Other polynomials of these families are chosen in such a way that they form an orthonormal basis in the space of all algebraic polynomials $\mathfrak{M}^{2n+2}$ of degree $\leq 2n+1$ endowed with a inner product and a metric

$$(u,v) = \int_{-1}^{1} uvdx, \|u\| = \sqrt{(u,u)},$$

respectively. In other words, $\mathfrak{M}^{2n+2}$ is a subspace of the space $L_2[-1,1]$.

Let's define the notations QS1(n) and QS2(n) for the first K_i° and second kinds $K_i^\cdot$, respectively, of quasispectral polynomials. In order to approximate boundary value problems with Dirichlet conditions [3] (conditions of the first kind) and QS2(n) – with Neumann conditions [4], we used the polynomials QS1(n) (of the first kind) and QS2(n) (of the second kind). By creating the corresponding Fourier series, we were able to resolve the Dirichlet and Neumann problems.

The interval $[-1,1]$ is chosen as the standard interval in which special classes of quasispectral polynomials are studied. The results obtained in the work by linear substitution are easily transferred to an arbitrary interval $[a, b]$. Also, these results will find application in spaces of higher dimensions. We limit ourselves to a simple case so as not to blur the essence of the question. We used these and similar results in the design and analysis of software for fast and effective solving of some problems of mathematical physics, for example works [2-4].

The goal of the paper is to investigate and analyze the approximation characteristics of one particular class, namely QS1, of algebraic polynomials designed for the approximation of functions that are solutions of boundary value problems in various metrics.

The starting point of the work was the classic Legendre polynomials $P_i = P_i(x), i = 0,1, \dots$ standardized by the condition $P_i(1) = 1$, orthogonal in L_2:

$$\int_{-1}^{1} P_i P_j dx = 0, i \neq j, \int_{-1}^{1} {P_i}^2 dx = \frac{2}{2i+1}$$

and corresponding Fourier-Legendre series:

$$f = \sum_{i=0}^{\infty} f_i P_i, f_i = \frac{2i+1}{2} \int_{-1}^{1} f(x) P_i(x) dx.$$

The operator for taking the partial sum of the Fourier-Legendre series has the form

$$\pi_N(f) = \sum_{i=0}^{N-1} f_i P_i$$

and as is well known[1], gives the best approximation in L_2:

$$\|f - \pi_N\| \leq \min_{p_N \in \mathfrak{M}^N} \|f - p_N\|.$$

The property of orthonormality of quasispectral polynomials provides the possibility of presenting this operator in two more forms:

$$\pi_N(f) = \sum_{i=0}^{N-1} f_i^\circ K_i^\circ \, , f_i^\circ = \int_{-1}^{1} f K_i^\circ dx \, , i = 1, \ldots, N,$$

$$\pi_N(f) = \sum_{i=0}^{N-1} f_i^\bullet K_i^\bullet \, , f_i^\bullet = \int_{-1}^{1} f K_i^\bullet dx \, , i = 1, \ldots, N.$$

The family of polynomials of the first kind QS1(n), which includes interior polynomials and two polynomials $K_{2n+1}^\circ, K_{2n+2}^\circ$, which are called boundary, is orthonormal in the metric of space $L_2[-1,1]$. Boundary polynomials are defined by formulas

$$K_{2n+2}^\circ = {\kappa_{2n+2}}^{-1} D P_{2n+2}, K_{2n+1}^\circ = {\kappa_{2n+1}}^{-1} D P_{2n+1}, \quad (3)$$

moreover

$$K_{2n+2}^\circ(1) = \frac{\kappa_{2n+2}}{2}, K_{2n+1}^\circ(1) = \frac{\kappa_{2n+1}}{2}, \kappa_i = \sqrt{i(i+1)}.$$

Interior quasispectral polynomials are determined from the condition of their transformation to zero at the ends of the main interval:

$$K_i^\circ(\pm 1) = 0, i = 1, \ldots, 2n.$$

For $i = 1, \ldots, n$ the following formulas for the differentiation of quasispectral polynomials, hold true

$$D^2 K_{2i}^\circ(x) = -\lambda_{2i}^\circ K_{2i}^\circ(x) + \tau_{2i}^\circ K_{2n+2}^\circ(x), \quad (4)$$

$$D^2 K_{2i-1}^\circ(x) = -\lambda_{2i-1}^\circ K_{2i-1}^\circ(x) + \tau_{2i-1}^\circ K_{2n+1}^\circ(x), \quad (5)$$

where

$$\lambda_i^\circ = -\int_{-1}^{1} \left(D^2 K_i^\circ(x)\right) K_i^\circ(x) \, dx = \int_{-1}^{1} \left(D K_i^\circ(x)\right)^2 dx > 0,$$

$$\tau_{2i-1}^\circ = \int_{-1}^{1} \left(D^2 K_{2i-1}^\circ(x)\right) K_{2n+1}^\circ(x) \, dx, \tau_{2i}^\circ = \int_{-1}^{1} \left(D^2 K_{2i}^\circ(x)\right) K_{2n+2}^\circ(x) \, dx. \quad (6)$$

We present the finite Fourier-Legend series of the given function $u \in L_2[-1,1]$ in the form

$$u^N = \sum_{i<N} u_i^\circ K_i^\circ(x) \quad (7)$$

and thereby we define the operator $\pi^N : L_2 \to \mathfrak{M}^N$ of taking the partial sum of the Fourier-Legendre series, but in the case of QS1(n) basis.

Taking into account the orthonormality of the polynomials of the family QS1(n) in the case of the classical Fourier-Legendre series, its coefficients are calculated by the formulas

$$u_i^\circ = \int_{-1}^{1} u(x) K_i^\circ(x) dx\,, i = 1, \ldots 2n+2. \quad (8)$$

The interior quasispectral polynomials become zero at the points ±1, and therefore, when the function $u \in H^1$, its boundary Fourier coefficients can be found approximately from the interpolation conditions:

$$u_{2n+1}^\circ = \frac{u(1)+u(-1)}{\sqrt{(2n+1)(2n+2)}}, u_{2n+2}^\circ = \frac{u(1)-u(-1)}{\sqrt{(2n+2)(2n+3)}} \quad (9)$$

and the resulting Fourier series will be called a modified Fourier series.

Let

$$\partial u^N = u_{2n+1}^\circ K_{2n+1}^\circ(x) + u_{2n+2}^\circ K_{2n+2}^\circ(x). \quad (10)$$

The interior quasispectral polynomials of the QS1 family are orthogonal in the Sobolev space H^1 (see (1)), therefore, the interior QS1 Fourier coefficients of the function $u \in H_0^1$, i.e., in the case of homogeneous conditions, we determine by the formula

$$u_i^\circ = \frac{(u, K_i^\circ)_1}{(K_i^\circ, K_i^\circ)_1}, \quad (11)$$

which actually corresponds to the expand of functions in the energy space metric. In the case of heterogeneous conditions, the interior QS1 Fourier coefficients are defined by the formulas

$$u_i^\circ = \frac{(u - \partial u^N, K_i^\circ)_1}{(K_i^\circ, K_i^\circ)_1}. \quad (12)$$

The question arises, is there not a big contradiction here in that the boundary coefficients are defined in (8) relative to the inner product induced by the space $L_2[-1,1]$ and the interior ones according to the formulas (11), (12), which correspond to the energy metric of the space H_0^1? The following lemma provides a partial answer.

Lemma 1. If the function $u \in H_0^1$ is a polynomial from $\mathfrak{M}^N$, then the interior QS1 Fourier coefficients are simultaneously the L_2–Fourier coefficients in the basis QS1(n):

$$u_i^\circ = \frac{(u - \partial u^N, K_i^\circ)_1}{(K_i^\circ, K_i^\circ)_1} = (u - \partial u^N, K_i^\circ), u \in \mathfrak{M}^N. \quad (13)$$

Proof. The lemma is established by checking formula on each element of the basis, in particular, the interior coefficients are calculated according to formula (12); in the case of boundary polynomials, that is, when $u = K^{\circ}_{2n+1}$ or $u = K^{\circ}_{2n+2}$, the boundary coefficients are equal to zero: $u^{\circ}_{2n+1} = 0, u^{\circ}_{2n+2} = 0$.

Remark. In the case of polynomial $u \in \mathfrak{M}^N$,

$$u = \partial u^N + \sum_{i=1}^{2n} u^{\circ}_i K^{\circ}_i, \qquad (14)$$

where the polynomial ∂u^N is given in the form (10), with the boundary coefficients determined either by formulas (8) or by formulas (9) (from the interpolation conditions), and we find the interior Fourier coefficients either by formulas (8) , or by formulas (12). Choosing one of the two Fourier coefficient formulas from the pair will produce different outcomes for an arbitrary function. In the case of, for instance, the approximate solution of boundary value problems for elliptic equations, using formulas of the interpolation type (9) to calculate the boundary coefficients and formulas of the QS1 type to calculate the interior coefficients (12) is a reasonable choice from the perspective of practical applications. As a result, we arrive at the following form of the approximation operator:

$$Q_N u = \partial u^N + Q^0_N u, Q^0_N u = \sum_{i=1}^{2n} u^{\circ}_i K^{\circ}_i,$$

$$\partial u^N = \frac{u(1)+u(-1)}{\sqrt{(N-1)N}} K^{\circ}_{N-1} + \frac{u(1)-u(-1)}{\sqrt{(N+1)N}} K^{\circ}_N, N = 2n+2$$

for any given function $u \in H^1$.

An arbitrary function $u \in H^1$ has the form

$$u = A_N u + \int_{-1}^{x} \sum_{i=N}^{\infty} u'_i P_i(s) ds \,, u'_i = \frac{2i+1}{2} \int_{-1}^{1} Du P_i(x) dx, \qquad (15)$$

where $A_N u$ is an approximation polynomial given by the formula

$$A_N u = \frac{u(1)+u(-1)}{2} + \frac{u(1)-u(-1}{2} x + \int_{-1}^{x} \sum_{i=1}^{2n} u'_i P_i(s) ds, \qquad (16)$$

and $A_N : H^1 \rightarrow \mathfrak{M}^N$ is the corresponding operator.

Each of the operators π_N, Q_N should be investigated from the point of view of the approximation error. In order to determine how much each operator differs from the others, it is important to look at each from the perspective of approximation error, or how dissimilar their results are from those of the approximation operator A_N. We shall examine their application to the arbitrary function $u \in H^1$ for this reason, which may always be expressed in the form (15). They are both projective, hence applying them to the polynomial $A_N u$ does not change it:

$$\pi_N A_N u = A_N u, Q_N A_N u = A_N u.$$

Therefore, we will apply these operators to the remainder of the series (15), namely

$$\pi_N \varepsilon_N u = \pi_N \int_{-1}^{x} \sum_{i=N}^{\infty} u'_i P_i(s) ds \, , Q_N \varepsilon_N u = Q_N \int_{-1}^{x} \sum_{i=N}^{\infty} u'_i P_i(s) ds.$$

By integrating the Legendre polynomials, we get

$$\varepsilon_N u = \sum_{i=N}^{\infty} u'_i \frac{P_{i+1} - P_{i-1}}{2i+1} ds,$$

hence

$$\pi_N \varepsilon_N u = -\frac{u'_{N-1}}{2N-1} P_{N-2} - \frac{u'_N}{2N+1} P_{N-1},$$

i.e.

$$\pi_N u - A_N u = -\frac{u'_{N-1}}{2N-1} P_{N-2} - \frac{u'_N}{2N+1} P_{N-1},$$

$$\|\pi_N u - A_N u\| \leq 2^{-1} N^{-1} E_N(Du). \quad (17)$$

Now let's do the same with the polynomial $\eta = Q_N u - A_N u, \eta(\pm 1) = 0$. Let,

$$g = u'_{N-1} J P_{N-1} + u'_N J P_N, J f(x) = \int_{-1}^{x} f(s) ds \, , f \in L_2. \quad (18)$$

Then,

$$Q_N^0 g = u'_{N-1} Q_N^0 J P_{N-1} + u'_N Q_N^0 J P_N,$$

where

$$Q_N^0 J P_{N-1} = \sum_{i=1}^{n} [J P_{N-1}]_{2i-1}^{\circ} K_{2i-1}^{\circ} \, , Q_N^0 J P_N = \sum_{i=1}^{n} [J P_N]_{2i}^{\circ} K_{2i}^{\circ},$$

$$[J P_{N-1}]_i^{\circ} = \frac{(J P_{N-1}, K_i^{\circ})}{\lambda_i^{\circ} + c}, [J P_N]_i^{\circ} = \frac{(J P_N, K_i^{\circ})}{\lambda_i^{\circ} + c}.$$

Based on (13) we get

$$\|Q_N^0 J P_{N-1}\|^2 = \sum_{i=1}^{n} [J P_{N-1}]_{2i-1}^{\circ}{}^2 \, , \|Q_N^0 J P_N\| = \sum_{i=1}^{n} [J P_N]_{2i}^{\circ}{}^2. \quad (19)$$

Let us now estimate the quantities (19). By the definition of the inner product in H^1 and (2), we have

$$(JP_N, K^\circ_{2i})_1 = (DJP_N, DK^\circ_{2i}) + \mathrm{c}(JP_N, K^\circ_{2i}) = \left(P_N, \sqrt{\lambda^\bullet_{2i}} K^\bullet_{2i}\right) + \mathrm{c}(JP_N, K^\circ_{2i}),$$

and so

$$(JP_N, K^\circ_{2i})_1 = \mathrm{c}(JP_N, K^\circ_{2i-1}), (JP_{N-1}, K^\circ_{2i-1})_1 = \mathrm{c}(JP_{N-1}, K^\circ_{2i-1}). \qquad (20)$$

Applying multiple integration, integration by parts and (4) ((5) in odd case), we get

$$(JP_N, K^\circ_{2i}) = (D^2 J^3 P_N, K^\circ_{2i}) = (J^3 P_N, D^2 K^\circ_{2i}) = (J^3 P_N, -\lambda^\circ_{2i} K^\circ_{2i} + \tau^\circ_{2i} K^\circ_{2n+2}),$$

and consequently

$$(JP_N, K^\circ_{2i}) = -\lambda^\circ_{2i}(J^3 P_N, K^\circ_{2i}) + \tau^\circ_{2i}(J^3 P_N, K^\circ_{2n+2}), N = 2n + 2. \qquad (21)$$

Next equality follows from (20) and (21):

$$[JP_N]^\circ_{2i} = \frac{(JP_N, K^\circ_{2i})_1}{\lambda^\circ_{2i} + c} = -\frac{\lambda^\circ_{2i}}{\lambda^\circ_{2i} + c}(J^3 P_N, K^\circ_{2i}) + \frac{\tau^\circ_{2i}}{\lambda^\circ_{2i} + c}(J^3 P_N, K^\circ_{2n+2}) \qquad (22)$$

Due to equality (3) and the Legendre polynomials' orthogonality, we obtain

$$(J^3 P_N, K^\circ_{2n+2}) = (J^3 P_N, {\kappa_N}^{-1} D P_N) = -(J^2 P_N, {\kappa_N}^{-1} P_N),$$

whence, repeatedly applying the formula for integration of Legendre polynomials

$$JP_i = (P_{i+1} - P_{i-1})/(2i + 1)\,, i = 1,2, \ldots,$$

we will get

$$(J^3 P_N, K^\circ_{\mathrm{N}}) = O(N^{-4}). \qquad (23)$$

Take into account formula (20) from [4]:

$$\sum_{i=1}^{n} \left(\frac{\tau^\circ_{2i}}{\lambda^\circ_{2i}}\right)^2 = \frac{n(2n+1)}{4n+3} = \frac{(N-1)(N-2)}{4N-2} = O(N),$$

and formula (23), we find

$$\|Q^0_N J P_N\| = \sqrt{\sum_{i=1}^{n} {[JP_N]^\circ_{2i}}^2} = O\left(N^{-3-1/2}\right),$$

Similarly, we will get an estimation

$$\|Q^0_N J P_{N-1}\| = \sqrt{\sum_{i=1}^{n} {[JP_{N-1}]^\circ_{2i-1}}^2} = O\left(N^{-3-1/2}\right).$$

From the last two estimations and (18), we obtain the asymptotic estimation

$$\|Q_N^0 \varepsilon_N\| = \|Q_N^0 g\| = \|u'_{N-1} JP_{N-1} + u'_N JP_N\| \le O(N^{-3}) E_N(Du).$$

In this way, we established the desired estimate of the deviation QS1 of the polynomial approximation $Q_N u$ of the function $u \in H^1$ from the approximation polynomial (16):

$$\|Q_N u - A_N u\| \le O(N^{-3}) E_N(Du). \tag{24}$$

Comparing two estimates: given and (17), we get

$$\|Q_N u - \pi_N u\| \le \|Q_N u - \pi_N u\| + \|Q_N u - A_N u\| \le \left(2^{-1} N^{-1} + O(N^{-3})\right) E_N(Du),$$

from which we obtain an important asymptotic estimation

$$\|Q_N u - \pi_N u\| \le 2^{-1} N^{-1} E_N(Du). \tag{25}$$

Conclusions. In the case of a modified Fourier series, estimate (25) provides an answer to the question of whether there will be a significant difference between calculating the interior Fourier coefficients of the function $u \in H^1$ precisely using the formulas for the inner product in the space L_2 or in the space H^1. It turned out that, from an approximation standpoint, there is no discernible difference in the metric L_2; more specifically, we obtain results that are almost identical, which is supported by the asymptotic estimate (25). In other words, the functions $u \in H^1$ are recovered with the highest order of approximation in the metric of the basis space L_2. According to the findings of works [3, 4], these findings apply to the situation of functions $H^1[-1,1]^d$, where d is the spatial dimension.

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